Ice-ocean boundary layers

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To model the ice-ocean boundary layer, we utilize the Navier-Stokes equations under the Boussinesq approximation. The governing equations can be written as follows:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{g}\rho + \mathbf{F}(z) \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \kappa_T \nabla^2 T \tag{3}$$

$$\frac{\partial S}{\partial t} + (\mathbf{u} \cdot \nabla)S = \kappa_S \nabla^2 S \tag{4}$$

The first equation represents the momentum equation. Here, $\mathbf{u} = \mathbf{u}(x, y, z)$ denotes the fluid velocity in m/s, ρ_0 is the reference density in kg/m^3, p is the pressure in Pa, and ν is the kinematic viscosity in m^2/s. The term $\mathbf{g}$ represents the gravitational acceleration in m/s^2, and ρ is the fluid density given by:

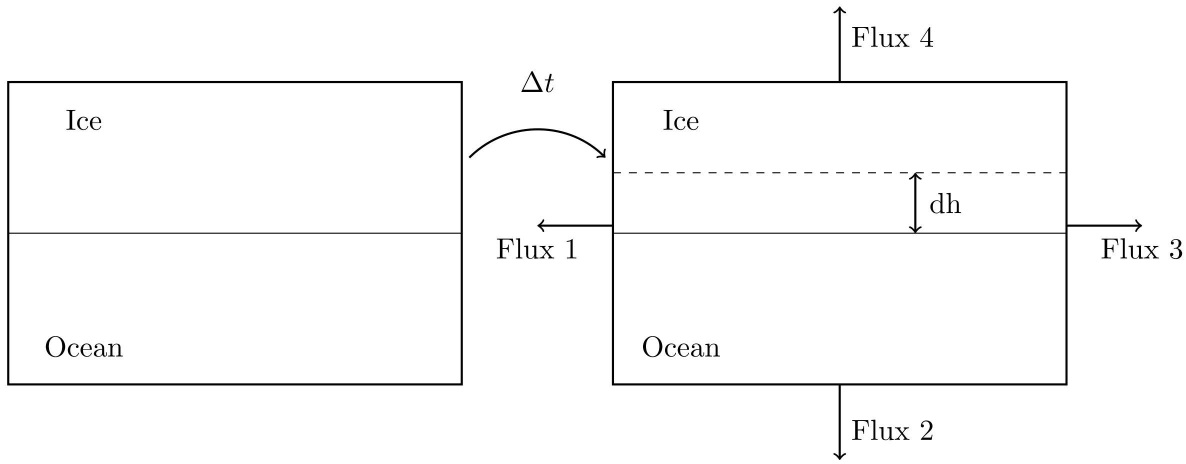
$$\rho = \rho_0 \left[1 - \beta_T(T - T_0) + \beta_S(S - S_0)\right]$$

In this expression:

- β_T is the thermal expansion coefficient in 1/K,
- β_S is the haline contraction coefficient in 1/(g/kg),
- T is the temperature in K,
- T_0 is the reference temperature in K,
- S is the salinity in g/kg,
- S_0 is the reference salinity in g/kg,

The second equation is the continuity equation, which ensures the incompressibility of the fluid. The third and fourth equations describe the transport of temperature and salinity, respectively, where κ_T is the thermal diffusivity and κ_S is the salinity diffusivity.

Ice-ocean boundary conditions To compute the boundary conditions, we evaluate the fluxes necessary to move the ice-ocean interface. When the ice is melting, freshwater is released into the ocean, leading to an increase in the ocean temperature at the upper boundary and a decrease in salinity.



The temperature transport equation is given by:

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \kappa_T \nabla^2 T \tag{5}$$

This can be rewritten as:

$$\frac{\partial T}{\partial t} + \nabla \cdot (\mathbf{u}T - \kappa_T \nabla T) = 0 \tag{6}$$

where $\mathbf{u}T$ represents the advective flux of temperature and $-\kappa_T \nabla T$ represents the diffusive flux of temperature.

Integrating over a control volume V with surface S, we get:

$$\int_V \left(\frac{\partial T}{\partial t} + \nabla \cdot (T\mathbf{u} - \kappa_T \nabla T) \right) dV = \mathbf{q} \cdot \mathbf{n} \tag{7}$$

Applying the divergence theorem:

$$\int_V \frac{\partial T}{\partial t} \, dV + \int_S (T\mathbf{u} - \kappa_T \nabla T) \cdot \mathbf{n} \, dS = \mathbf{q} \cdot \mathbf{n} \tag{8}$$

where $\mathbf{n}$ is the unit normal vector to the surface S and $\mathbf{q}$ is the heat flux vector (W/m^2). For mass conservation, the first term in the equation is zero. We can decompose the second term over the four surfaces, each corresponding to the fluxes in the previous scheme. We impose periodic boundary conditions on surfaces 1 and 3, which results in their fluxes canceling out. Additionally, on the impermeable wall at the bottom (surface 2), the flux is zero. The only term that remains is:

$$\int_S (T\mathbf{u} - \kappa_T \nabla T) \cdot \mathbf{n} \, dS_4 = (\mathbf{q} \cdot \mathbf{n}) \tag{9}$$

This integral can be expressed as:

$$\int_S (T\mathbf{u} - \kappa_T \nabla T) \, dx \, dy \, \mathbf{e}_z = \frac{L_x L_y}{\langle L_x \rangle \langle L_y \rangle} (u_z T - \kappa_T \partial_z T) \tag{10}$$

Since $u_z = 0$, the only term that remains is $-\kappa_T \partial_z T$. This term can be compared to the heat required to change the ice-ocean interface. At the ice-ocean interface, the phase change due to melting or freezing is associated with the heat flux across the boundary as:

$$\mathbf{q} \cdot \mathbf{n} = L_f C_p^{-1} \frac{\partial h}{\partial t} \tag{11}$$

where Ł_f is the latent heat of fusion (J/kg) and h is the ice thickness loss (m). For melting, $L_f C_p^{-1} \frac{\partial h}{\partial t}$ represents the heat required to change the phase of the ice. Thus, the heat boundary condition at the ice-ocean interface can be expressed as:

$$-\kappa_T \partial_z T = L_f C_p^{-1} \frac{\partial h}{\partial t} \tag{12}$$

We can follow a similar approach for salinity. At the ice-ocean interface, the phase change due to melting or freezing is associated with the salt flux across the boundary, which is given by:

$$\mathbf{q} \cdot \mathbf{n} = S \frac{\partial h}{\partial t} \tag{13}$$

where SS is the salinity of the ocean. Thus, the salt boundary condition at the ice-ocean interface can be expressed as:

$$-\kappa_S \partial_z S = S \frac{\partial h}{\partial t} \tag{14}$$

Equating both boundary conditions, we find:

$$\partial_z S = \left(\frac{\kappa_T}{\kappa_S} \right) \left(\frac{C_p}{L_f} \right) S \partial_z T \tag{15}$$

Top Boundary Conditions

Ice

$$\partial_z S = (\kappa_T / \kappa_S)(C_p / L_f)\, S\, \partial_z T$$

$$T = f(S)$$

Ocean

Bottom Boundary Conditions

$$S = S_\infty$$

$$T = T_\infty$$