Ice-ocean boundary layers

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To model the ice-ocean boundary layer, we utilize the Navier-Stokes equations under the Boussinesq approximation. In the bulk, the governing equations can be written as follows:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{g}\rho + \mathbf{F}(z) \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \kappa_T \nabla^2 T \tag{3}$$

$$\frac{\partial S}{\partial t} + (\mathbf{u} \cdot \nabla)S = \kappa_S \nabla^2 S \tag{4}$$

The first equation represents the momentum equation. Here, $\mathbf{u} = \mathbf{u}(x, y, z)$ denotes the fluid velocity in m/s, ρ_0 is the reference density in kg/m^3, p is the pressure in Pa, and ν is the kinematic viscosity in m^2/s. The term $\mathbf{g}$ represents the gravitational acceleration in m/s^2, and ρ is the fluid density given by:

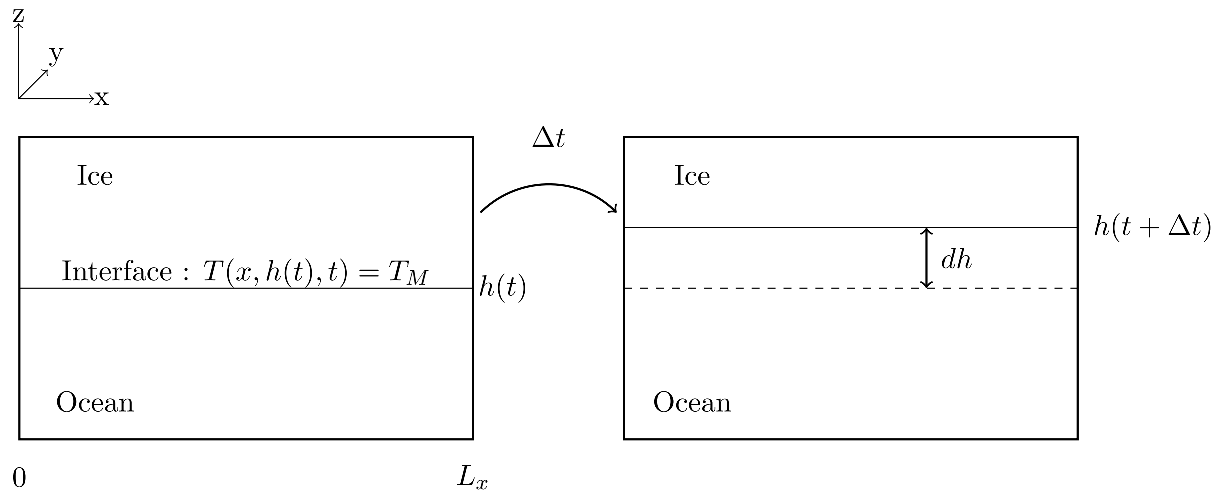
$$\rho = \rho_0 \left[1 - \beta_T(T - T_0) + \beta_S(S - S_0)\right]$$

In this expression:

- β_T is the thermal expansion coefficient in 1/K,
- β_S is the haline contraction coefficient in 1/(g/kg),
- T is the temperature in K,
- T_0 is the reference temperature in K,
- S is the salinity in g/kg,
- S_0 is the reference salinity in g/kg,

The second equation is the continuity equation, which ensures the incompressibility of the fluid. The third and fourth equations describe the transport of temperature and salinity, respectively, where κ_T is the thermal diffusivity and κ_S is the salinity diffusivity.

Ice-ocean boundary conditions Our setup assumes a two-dimension flow with a homogeneous ice-ocean interface. The temperature at this interface is equal to the melting temperature (T_M). We also assume that this interface moves with a velocity equal to $u_z\Big|_{z=h(t)} = \dot{h}(t)$.



To describe the boundary conditions at the ice-ocean interface, we calculate the internal energy of the water.

$$E_i(t) = C_p \int_0^{L_x} dx \int_0^{h(t)} dz\, T(x, z, t) \tag{5}$$

where $T(x, z, t)$ is the seawater temperature and C_p is the seawater heat capacity ($J/(kgK)$). When the ice is melting, the ice thickness decreases and freshwater is released into the ocean, leading to an increase in the internal energy. This variation is given by:

$$\frac{d}{dt} E_i(t) = C_p \int_0^{L_x} dx\, T(x, h(t), t)\, \dot{h}(t) + C_p \int_0^{L_x} dx \int_0^{h(t)} dz\, \partial_t T(x, z, t) = L_f L_x u_z \tag{6}$$

where L_f is the latent heat of fusion (J/kg). We can write this expresion as:

$$\int_0^{L_x} dx \int_0^{h(t)} dz\, \partial_t T(x, z, t) = \frac{L_f}{C_p} L_x u_z - \int_0^{L_x} dx\, T(x, h(t), t)\, \dot{h}(t) \tag{7}$$

Therefore,

$$\int_0^{L_x} dx \int_0^{h(t)} dz\, \partial_t T(x, z, t) = \frac{L_f}{C_p} L_x u_z - L_x T_M \dot{h}(t) \tag{8}$$

On the other hand, the temperature transport equation at the ice boundary is given by:

$$\frac{\partial T}{\partial t} + \nabla \cdot (\mathbf{u}T - \kappa_T \nabla T) = 0 \tag{9}$$

where $\mathbf{u}T$ represents the advective flux of temperature and $-\kappa_T \nabla T$ represents the diffusive flux of temperature. Integrating over a control volume V with surface S, we get:

$$\int_V \left(\frac{\partial T}{\partial t} + \nabla \cdot (T\mathbf{u} - \kappa_T \nabla T) \right) dV = 0 \tag{10}$$

Applying the divergence theorem:

$$\int_V \frac{\partial T}{\partial t}\, dV + \int_S (T\mathbf{u} - \kappa_T \nabla T) \cdot \mathbf{n}\, dS = 0 \tag{11}$$

where $\mathbf{n}$ is the unit normal vector to the surface S. For a control volume where the surface S is aligned with the z-axis, the surface integral becomes:

$$\int_V \frac{\partial T}{\partial t}\, dV + \int_0^{L_x} dx\, (T\mathbf{u} - \kappa_T \nabla T) \cdot \mathbf{e}_z = 0 \tag{12}$$

Thus,

$$\int_V \frac{\partial T}{\partial t}\, dV + L_x T_M u_z - L_x \kappa_T \partial_z T = 0 \tag{13}$$

Replacing the term $\int_V \frac{\partial T}{\partial t}\, dV$ with the previously derived result and $u_z = \dot{h}(t)$, the expression follows as:

$$\frac{L_f}{C_p} L_x u_z - L_x T_M \dot{h}(t) + L_x T_M \dot{h}(t) - L_x \kappa_T \partial_z T = 0 \tag{14}$$

Finally, the heat boundary condition at the ice-ocean interface can be expressed as:

$$-\kappa_T \partial_z T = \frac{L_f}{C_p} \dot{h}(t) \tag{15}$$

We can follow a similar approach for salinity. At the ice-ocean interface, the phase change due to melting or freezing is associated with the salt flux across the boundary, which is given by:

$$-\kappa_S \partial_z S = S \frac{\partial h}{\partial t} \tag{16}$$

where S is the salinity of the ocean.

Equating both boundary conditions, we find:

$$\partial_z S = \left(\frac{\kappa_T}{\kappa_S}\right) \left(\frac{C_p}{L_f}\right) S \partial_z T \tag{17}$$

Top Boundary Conditions

Ice

$\partial_z S = (\kappa_T / \kappa_S)(C_p / L_f)\, S\, \partial_z T$

$T = T_M = T(S, P_b)$

Ocean

Bottom Boundary Conditions

$S = S_\infty$

$T = T_\infty$